Earth Coordinates

CESAR's Booklet

It's the purpose of this booklet to describe how to use Earth coordinates and to provide some tools that may come in handy when working with them.

Earth Coordinates

Locate a point in Earth

As you know, the Earth is a sphere, or at least it's very similar to a sphere. As a sphere is a 3D object, we'll need three coordinates to describe where a point in Earth is located. There are many ways of doing this, but in this booklet we'll only explain the most extended method:

The first coordinate is height, that is, how higher (or lower) than the sea surface is our point. Once we know the height, the only thing left is describe where in the 2D surface of the sphere the point is, then by adding the height we'll have the exact location.

(To follow the next paragraphs where how to locate a point in Earth's surface is described, please, do first take a look at Image 1 and Image 2.)

In the same way we had the sea level for measuring the height, we'll need a reference for positioning our point in the Earth surface. To choose this reference we will use the Earth's axis. The points where the Earth's axis crosses the Earth's surface are the North Pole and the South Pole. You can draw in Earth's surface as many straight lines as you want that go from one pole to an other. That lines are called meridians and can be seen in the front-page-image. Once you have the meridians, you can draw the other lines that are seen in the image, those lines are perpendicular to the meridians and to the Earth's axis, and are called parallels.

The parallel that is at the same distance from both poles is the equator. The angle centred in Earth's centre that goes straight from the equator to a point's parallel is called latitude. This angle is considered positive in the northern hemisphere and negative in the southern one. That means that equator is considered to be at 0º latitude, north pole at 90º and south pole at -90º.

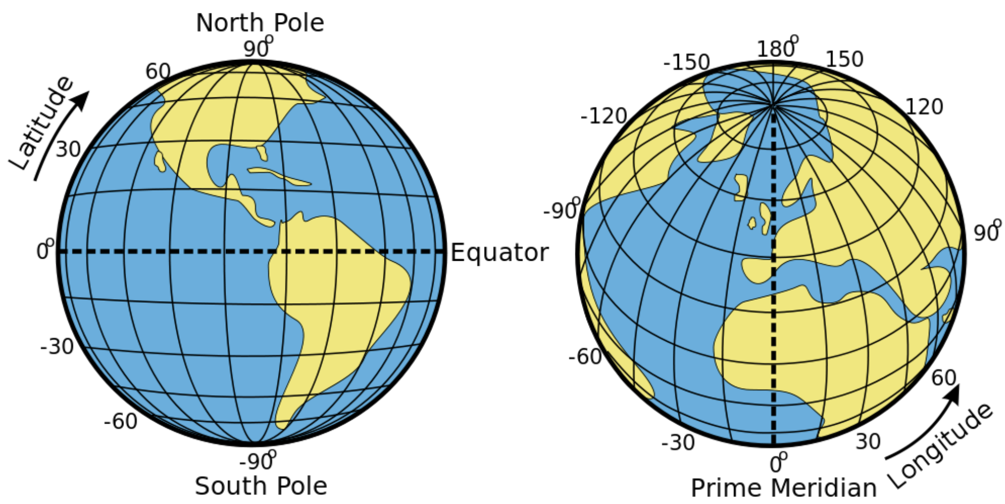

Image 1: Earth Coordinates

The meridian that passes through the Royal Observatory in Greenwich is called the prime meridian. The angle centred in Earth's centre and located in the equatorial plane, that goes straight from the prime meridian to the meridian of a point is called longitude. This angle is considered positive if it grows counter-clockwise as viewed from the north pole and negative otherwise. This means that the prime meridian is considered to be at 0° longitude and its opposite meridian is 180° longitude.

The latitude and longitude angles are enough to place a point in the Earth's surface and along with height give the exact location of a point in Earth. Note that when used in math, latitude is expressed by φ and longitude by λ.

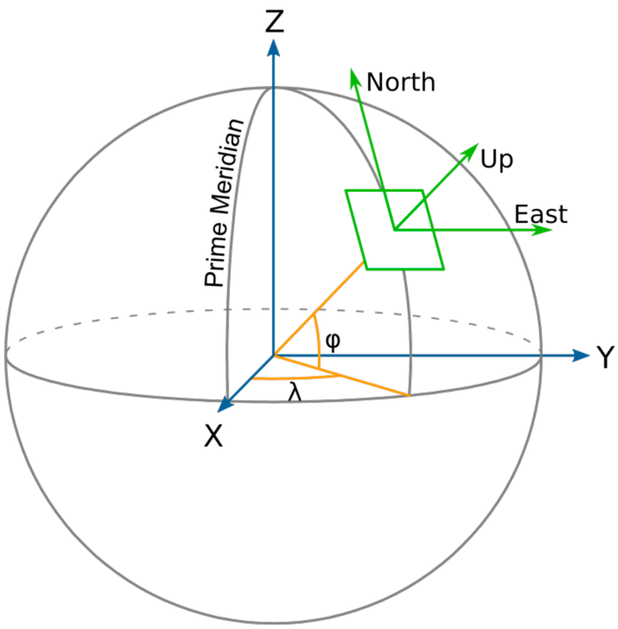

Image 2: Earth Geometry

Latitude and longitude nomenclature

Latitude and longitude angles can be expressed in different ways. Each angle may be expressed as a decimal or as a sexagesimal degree. For example, ESAC's latitude is 40.444° or 40° 26' 40''. Also the sign of the angle that can be expressed with a traditional minus sign if negative and no sign at all if positive, can also be written in an other way. That is using N for positive latitude angles, S for negative ones, E for positive longitude angles, and W for negative ones. For example, ESAC's longitude is -3.953° or 3.953 W.

This means that a coordinate can be expressed in four ways. Using the same example as before, ESAC location can be expressed as

40.000°, -3.953°.   40.000 N, 3.953W.   40° 26' 40'', -3° 57' 9''.   40° 26' 40'' N, 3° 57' 9'' W.

The last one us usually the most common, but the first one is the one that is to be used in math.

Distances between coordinates

Now that it's clear how to describe the position of points in Earth, let's say you want to calculate the distance between two of them. First thing we should ask ourselves is: what distance? The straight-line distance or the distance if traveling along Earth's surface? Each calculation is different. Let's start with the straight-line distance:

Straight-line distance

Our starting point is the Euclidian distance between two points in 3D Cartesian coordinates, or in other words the known-by-all formula

$$d = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

that may also be expressed as

$$d = \left(x^2 + x'^2 - 2xx' + y^2 + y'^2 - 2yy' + z^2 + z'^2 - 2zz'\right)^{1/2}$$

Now we'll just change from Cartesian coordinates to Earth coordinates. Looking at the previous image it's clear that this change is

$$x = r \cdot \cos(\varphi) \cdot \cos(\lambda)$$

$$y = r \cdot \cos(\varphi) \cdot \sin(\lambda)$$

$$z = r \cdot \sin(\varphi)$$

which is very similar to spherical coordinates' change.

After changing coordinates, the previous equation looks like this:

$$d = ((r \cdot \cos(\varphi) \cdot \cos(\lambda))^2 + (r' \cdot \cos(\varphi') \cdot \cos(\lambda'))^2 - 2 \cdot r \cdot \cos(\varphi) \cdot \cos(\lambda) \cdot r' \cdot \cos(\varphi') \cdot \cos(\lambda')$$

$$+(r \cdot \cos(\varphi) \cdot \sin(\lambda))^2 + (r' \cdot \cos(\varphi') \cdot \sin(\lambda'))^2 - 2 \cdot r \cdot \cos(\varphi) \cdot \sin(\lambda) \cdot r' \cdot \cos(\varphi') \cdot \sin(\lambda')$$

$$+(r \cdot \sin(\varphi))^2 + (r' \cdot \sin(\varphi'))^2 - 2 \cdot r \cdot \sin(\varphi) \cdot r' \cdot \sin(\varphi'))^{1/2}$$

Providing we know their coordinates, this formula allows us to calculate the distance between two points in Earth. But it is so messy that it's not really a useful tool. Let's give it a better look. For doing that we'll forget about height and assume that both r and r' are just Earth's radius. We know can express the equation as

$$d = R_E \cdot (cos^2(\varphi) \cdot cos^2(\lambda) + cos^2(\varphi') \cdot cos^2(\lambda') - 2 \cdot \cos(\varphi) \cdot \cos(\lambda) \cdot \cos(\varphi') \cdot \cos(\lambda')$$
$$+cos^2(\varphi) \cdot sin^2(\lambda) + cos^2(\varphi') \cdot sin^2(\lambda') - 2 \cdot \cos(\varphi) \cdot \sin(\lambda) \cdot \cos(\varphi') \cdot \sin(\lambda')$$
$$+ sin^2(\varphi) + sin^2(\varphi') - 2 \cdot \sin(\varphi) \cdot \sin(\varphi'))^{1/2}$$

that already looks better.

Now we are taking the equation

$$d = R_E \cdot (cos^2(\varphi) \cdot cos^2(\lambda) + cos^2(\varphi') \cdot cos^2(\lambda') - 2 \cdot \cos(\varphi) \cdot \cos(\lambda) \cdot \cos(\varphi') \cdot \cos(\lambda')$$
$$+cos^2(\varphi) \cdot sin^2(\lambda) + cos^2(\varphi') \cdot sin^2(\lambda') - 2 \cdot \cos(\varphi) \cdot \sin(\lambda) \cdot \cos(\varphi') \cdot \sin(\lambda')$$
$$+ sin^2(\varphi) + sin^2(\varphi') - 2 \cdot \sin(\varphi) \cdot \sin(\varphi'))^{1/2}$$

and solving the positive and the negative parts separately. After solving them we'll just do

$$d = R_E \cdot (positive - negastive)^{1/2}$$

Let's start with the positive part

$$cos^2(\varphi)\, cos^2(\lambda) + cos^2(\varphi') cos^2(\lambda') + cos^2(\varphi) sin^2(\lambda) + cos^2(\varphi')\, sin^2(\lambda') + sin^2(\varphi) + sin^2(\varphi')$$

We can rearrange it as

$$cos^2(\varphi) \cdot \left(cos^2(\lambda) + sin^2(\lambda)\right) + cos^2(\varphi') \cdot (cos^2(\lambda') + sin^2(\lambda')) + sin^2(\varphi) + sin^2(\varphi')$$

But as you know

$$cos^2(x) + sin^2(x) = 1$$

so we can simplify the formula to

$$cos^2(\varphi) + cos^2(\varphi') + sin^2(\varphi) + sin^2(\varphi')$$

and then

$$cos^2(\varphi) + sin^2(\varphi) + cos^2(\varphi') + sin^2(\varphi') = 1 + 1 = 2$$

The positive part turned out to be just a single number. Lets now proceed with the negative part.

The starting point is

$$2 \cdot \cos(\varphi) \cdot \cos(\lambda) \cdot \cos(\varphi') \cdot \cos(\lambda') + 2 \cdot \cos(\varphi) \cdot \sin(\lambda) \cdot \cos(\varphi') \cdot \sin(\lambda') + 2 \cdot \sin(\varphi) \cdot \sin(\varphi')$$

which can be rearranged to

$$2 \cdot \cos(\varphi) \cdot \cos(\varphi') \cdot (\cos(\lambda) \cdot \cos(\lambda') + \sin(\lambda) \cdot \sin(\lambda')) + 2 \cdot \sin(\varphi) \cdot \sin(\varphi')$$

Time for another trigonometric relation, remember that

$$cos(a-b) = \cos(\mathrm{a})\cos(\mathrm{b}) + \sin(\mathrm{a})\sin(\mathrm{b})$$

so the previous formula turns to be just

$$2 \cdot \cos(\varphi) \cdot \cos(\varphi') \cdot \cos(\lambda - \lambda') + 2 \cdot \sin(\varphi) \cdot \sin(\varphi')$$

One last trigonometric relation and the job is done, you must know that

$$sin^2\left(\frac{x}{2}\right) = \frac{1}{2} \cdot (1 - \cos(x))$$

that can also be expressed as

$$\cos(x) = 1 - 2 \cdot sin^2\left(\frac{x}{2}\right)$$

Knowing this, our formula turns to

$$2 \cdot \cos(\varphi) \cdot \cos(\varphi') \cdot \left(1 - 2 \cdot sin^2\left(\frac{\lambda - \lambda'}{2}\right)\right) + 2 \cdot \sin(\varphi) \cdot \sin(\varphi')$$

and rearranging it

$$2 \cdot \cos(\varphi) \cdot \cos(\varphi') - \cos(\varphi) \cdot \cos(\varphi') \cdot 2 \cdot 2 \cdot sin^2\left(\frac{\lambda - \lambda'}{2}\right) + 2 \cdot \sin(\varphi) \cdot \sin(\varphi')$$

or also

$$-4 \cdot \cos(\varphi) \cdot \cos(\varphi') \cdot sin^2\left(\frac{\lambda - \lambda'}{2}\right) + 2 \cdot \cos(\varphi) \cdot \cos(\varphi') + 2 \cdot \sin(\varphi) \cdot \sin(\varphi')$$

It looks like we can use the two last relations once again

$$-4 \cdot \cos(\varphi) \cdot \cos(\varphi') \cdot sin^2\left(\frac{\lambda - \lambda'}{2}\right) + 2 \cdot \cos(\varphi - \varphi')$$

and then

$$-4 \cdot \cos(\varphi) \cdot \cos(\varphi') \cdot sin^2\left(\frac{\lambda - \lambda'}{2}\right) + 2 \cdot \left(1 - 2 \cdot sin^2\left(\frac{\varphi - \varphi'}{2}\right)\right)$$

that can be rearranged to

$$-4 \cdot \cos(\varphi) \cdot \cos(\varphi') \cdot sin^2\left(\frac{\lambda - \lambda'}{2}\right) - 4 \cdot sin^2\left(\frac{\varphi - \varphi'}{2}\right) + 2$$

Time to put the negative and the positive parts together

$$d = R_E \cdot \left(2 - \left(-4 \cdot \cos(\varphi) \cdot \cos(\varphi') \cdot sin^2\left(\frac{\lambda - \lambda'}{2}\right) - 4 \cdot sin^2\left(\frac{\varphi - \varphi'}{2}\right) + 2\right)\right)^{1/2}$$

that can be finally be written as

$$d = 2 \cdot R_E \cdot \sqrt{sin^2\left(\frac{\varphi - \varphi'}{2}\right) + \cos(\varphi) \cdot \cos(\varphi') \cdot sin^2\left(\frac{\lambda - \lambda'}{2}\right)}$$

which does look like a useful tool to calculate the straight-line distance between two coordinates.