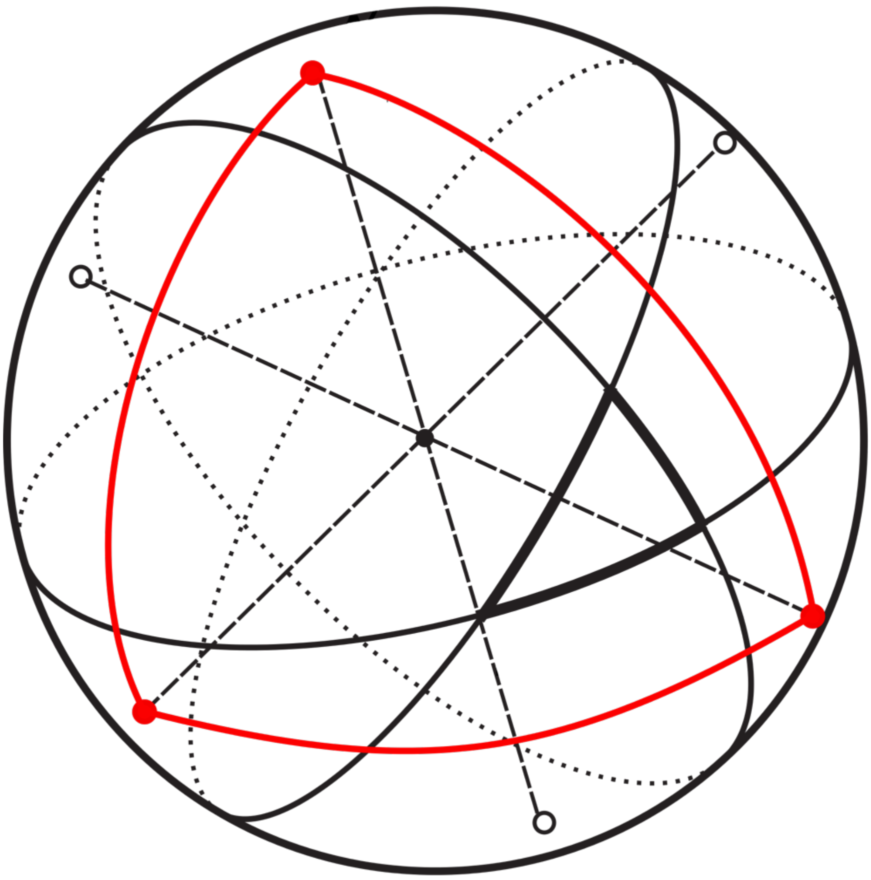
Spherical Trigonometry

CESAR's Booklet



It’s the purpose of this booklet to deduce some formulae and equations that may come in handy when working with spherical trigonometry.

The cosine-formula

Math Notation

$$\exists A,B,C,O \quad \exists \alpha \circledcirc O : A,B,C \in \alpha \quad \exists \beta \perp \overline{OA} : A \in \alpha,\beta \quad \exists D \in \beta,\overline{OB} \quad \exists E \in \beta,\overline{OC}$$

$$\exists \lambda := \widehat{AOB} \quad \exists \mu := \widehat{AOC} \quad \exists \omega := \widehat{BOC} \quad \exists \vartheta := \widehat{BAC} \quad \exists \kappa := \widehat{BCA} \quad \exists \eta := \widehat{CBA}$$

$$\therefore$$

$$\left|\overleftrightarrow{AD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \tan(\lambda) \quad \left|\overleftrightarrow{OD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \sec(\lambda) \quad \left|\overleftrightarrow{AE}\right| = \left|\overleftrightarrow{OA}\right| \cdot \tan(\mu) \quad \left|\overleftrightarrow{OE}\right| = \left|\overleftrightarrow{OA}\right| \cdot \sec(\mu)$$

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{AD}\right|^2 + \left|\overleftrightarrow{AE}\right|^2 - 2 \cdot \left|\overleftrightarrow{AD}\right| \cdot \left|\overleftrightarrow{AE}\right| \cdot \cos(\vartheta) \quad \left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OD}\right|^2 + \left|\overleftrightarrow{OE}\right|^2 - 2 \cdot \left|\overleftrightarrow{OD}\right| \cdot \left|\overleftrightarrow{OE}\right| \cdot \cos(\omega)$$

$$\therefore$$

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OA}\right|^2 \left[\tan^2(\lambda) + \tan^2(\mu) - 2 \cdot \tan(\lambda) \cdot \tan(\mu) \cdot \cos(\vartheta)\right]$$

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OA}\right|^2 \left[\sec^2(\lambda) + \sec^2(\mu) - 2 \cdot \sec(\lambda) \cdot \sec(\mu) \cdot \cos(\omega)\right]$$

$$\therefore$$

$$-2 \cdot \tan(\lambda) \cdot \tan(\mu) \cdot \cos(\vartheta) = 1 + 1 - 2 \cdot \sec(\lambda) \cdot \sec(\mu) \cdot \cos(\omega)$$

$$\therefore$$

$$\cos(\omega) = \cos(\mu) \cdot \cos(\lambda) + \sin(\mu) \cdot \sin(\lambda) \cdot \cos(\vartheta)$$

$$\square$$

Explanation

In case you need it, in this section we will explore in detail the previous deduction, so that the path to the cosine-formula it's clear and the formula itself is understandable:

First, you should know, that in geometry, it's common to name points with capital letters, while lines are usually named with lowercase letters and Greek letters are commonly reserved for angles and planes.

Also, before reading the math above, you must have a basic knowledge about math symbols:

$\exists$	There exists
◎	Concentric
:	Such that
$\in$	Element of
$\perp$	Perpendicular
:=	Definition
$\therefore$	Therefore
$\square$	End of proof

Now that you have this fundamental knowledge, lets start to read the math notation again. (It's a good idea to draw everything that you read, like in Image 1)

$\exists A,B,C,O$

This first statement, just means, that the deduction of the cosine-formula starts with the existence of four points in the space, named A, B, C and O, about which we know nothing yet.

$\exists \alpha \circledcirc O : A,B,C \in \alpha$

To this, it follows the existence of α that is concentric to O. α is a sphere, cantered in the point O. The sphere is defined so that the points A, B and C are part of it. So this far, we have a sphere, which centre is named O and three points in the sphere named A, B and C.

$\exists \beta \perp \overline{OA} : A \in \alpha, \beta$

Now a plane β appears. We know that the plane is perpendicular to the line that joins O and A. The line that joins O and A is just the radio of the sphere, going from the centre to the point A, so we just have a plain perpendicular to the radio. Then we find out that the point A is contained in both the sphere and the plane, so if the point A was part of the sphere and now is part of the plane too, then the plane perpendicular to the radio, must be at the surface of the sphere, touching the sphere only at the point A. That's the only way to have the point A both in the sphere and the plane.

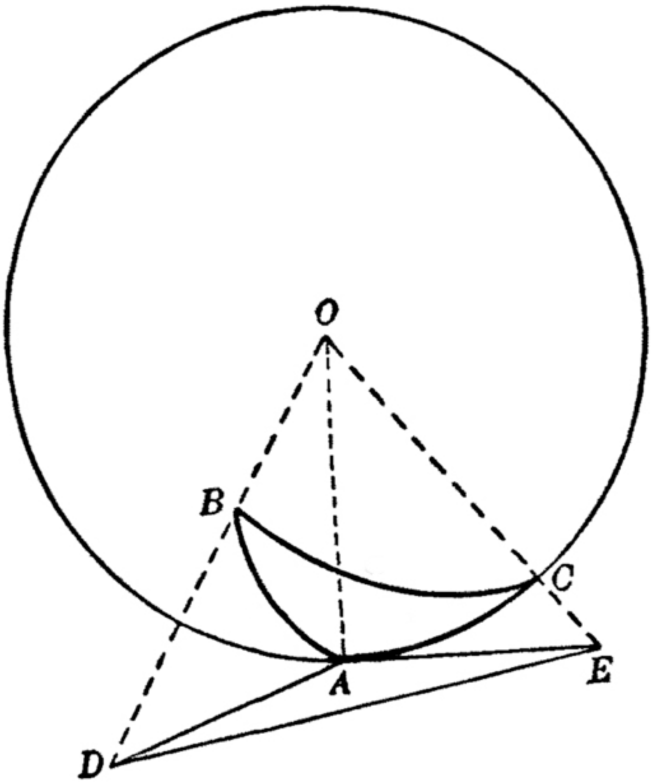


Image 1: Visual representation of the described situation

$\exists D \in \beta, \overline{OB}$

A new point D appears. The point is in the new plane β, and is also in the line that goes from O to B. So if we extend the line that goes from O to B until it intersects the plane, we will find the new point D.

$\exists E \in \beta, \overline{OC}$

The same explanation is repeated here. A new point E appears. The point is in the new plane β, and is also in the line that goes from O to C. So if we extend the line that goes from O to C until it intersects the plane, we will find the new point E.

$\exists\lambda := \widehat{AOB} \quad \exists\mu := \widehat{AOC} \quad \exists\omega := \widehat{BOC} \quad \exists\vartheta := \widehat{BAC} \quad \exists\kappa := \widehat{BCA} \quad \exists\eta := \widehat{CBA}$

This line is really easy, where just naming angles here. λ is the angle between A, O and B, so it's the angle located at O defined by the lines going from A to O and from B to O. You can apply the same to the other 5 angles with their new Greek-letter-names.

At this point we have explained the first two lines from the math-notation deduction, and the result is a picture like the one in the previous page, with its main elements properly named. After this two lines a therefore symbol is placed, meaning that the following lines can be deduced from the ones we already explained. As we represented the information from the first lines in a picture, we will be able to deduce the following just by looking at the picture we made. Let's get started:

$$\left|\overleftrightarrow{AD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \tan(\lambda)$$

Look carefully at the triangle between D, O and A, its straightforward that

$$\lambda := \widehat{AOB} \cong \widehat{AOD}$$

Knowing this we can deduce the basic trigonometric relations for the triangle

$$\overleftrightarrow{OA} = \cos(\lambda) \cdot \overleftrightarrow{OD}$$

and

$$\overleftrightarrow{AD} = \sin(\lambda) \cdot \overleftrightarrow{OD}$$

Now if we clear the line OD from the first relation and introduce it into the second, we obtain

$$\left|\overleftrightarrow{AD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \tan(\lambda)$$

which is what we saw in the math-notation section.

□

One formula done, five to go.

$$\left|\overleftrightarrow{OD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \sec(\lambda)$$

This one is basically the same as

$$\overleftrightarrow{OA} = \cos(\lambda) \cdot \overleftrightarrow{OD}$$

which we have already seen before.

□

$$\left|\overleftrightarrow{AE}\right| = \left|\overleftrightarrow{OA}\right| \cdot \tan(\mu)$$

This one, is no different to the first one

$$\left|\overleftrightarrow{AD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \tan(\lambda)$$

but looking at the AOD triangle instead of at the AOE.

□

$$\left|\overleftrightarrow{OE}\right| = \left|\overleftrightarrow{OA}\right| \cdot \sec(\mu)$$

And this one is the same as the previous one,

$$\left|\overleftrightarrow{OD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \sec(\lambda)$$

but looking at the AOE triangle instead of at the AOD.

□

The four equations in the first line should be now clear. Now only the two equations in the second line are left.

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{AD}\right|^2 + \left|\overleftrightarrow{AE}\right|^2 - 2 \cdot \left|\overleftrightarrow{AD}\right| \cdot \left|\overleftrightarrow{AE}\right| \cdot \cos(\vartheta)$$

This is simply the result of doing the vector operation

$$\overleftrightarrow{DE}^2 = \left(\overleftrightarrow{AD} - \overleftrightarrow{AE}\right)^2$$

which comes from

$$\overleftrightarrow{DE} = \overleftrightarrow{AD} - \overleftrightarrow{AE}$$

that is straightforward from the picture.

□

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OD}\right|^2 + \left|\overleftrightarrow{OE}\right|^2 - 2 \cdot \left|\overleftrightarrow{OD}\right| \cdot \left|\overleftrightarrow{OE}\right| \cdot \cos(\omega)$$

And clearly you can get to this other expression using the same procedure as before, the only difference is that in this case the original relation was

$$\overleftrightarrow{DE} = \overleftrightarrow{OD} - \overleftrightarrow{OE}$$

□

And with this last one, we already have deduced the five equations that came after the therefore sign. But now a new therefore sign comes, and two equations follow, this meaning that we should be able to deduce these two new equations from the previous six. Let's try:

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OA}\right|^2 \left[\tan^2(\lambda) + \tan^2(\mu) - 2 \cdot \tan(\lambda) \cdot \tan(\mu) \cdot \cos(\vartheta)\right]$$

This is exactly the same as

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OA}\right|^2 \cdot \tan^2(\lambda) + \left|\overleftrightarrow{OA}\right|^2 \cdot \tan^2(\lambda)\mu - 2 \cdot \left|\overleftrightarrow{OA}\right| \cdot \tan(\lambda) \cdot \left|\overleftrightarrow{OA}\right| \cdot \tan(\mu) \cdot \cos(\vartheta)$$

which if you look closely, it's the combination of three of the equations from the previous sections, just take

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{AD}\right|^2 + \left|\overleftrightarrow{AE}\right|^2 - 2 \cdot \left|\overleftrightarrow{AD}\right| \cdot \left|\overleftrightarrow{AE}\right| \cdot \cos(\vartheta)$$

replace

$$\left|\overleftrightarrow{AD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \tan(\lambda)$$

and

$$\left|\overleftrightarrow{OD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \sec(\lambda)$$

and you've got it.

□

After, this, it's probably clear that

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OA}\right|^2 \left[\sec^2(\lambda) + \sec^2(\mu) - 2 \cdot \sec(\lambda) \cdot \sec(\mu) \cdot \cos(\omega)\right]$$

can be obtained with the same procedure from

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OD}\right|^2 + \left|\overleftrightarrow{OE}\right|^2 - 2 \cdot \left|\overleftrightarrow{OD}\right| \cdot \left|\overleftrightarrow{OE}\right| \cdot \cos(\omega)$$

,

$$\left|\overleftrightarrow{OD}\right| = \left|\overleftrightarrow{OA}\right| \cdot \sec(\lambda)$$

and

$$\left|\overleftrightarrow{OE}\right| = \left|\overleftrightarrow{OA}\right| \cdot \sec(\mu)$$

□

Now, we can join this two equations two deduce the next one. If we look at

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OA}\right|^2 \left[\sec^2(\lambda) + \sec^2(\mu) - 2 \cdot \sec(\lambda) \cdot \sec(\mu) \cdot \cos(\omega)\right]$$

and

$$\left|\overleftrightarrow{DE}\right|^2 = \left|\overleftrightarrow{OA}\right|^2 \left[\tan^2(\lambda) + \tan^2(\mu) - 2 \cdot \tan(\lambda) \cdot \tan(\mu) \cdot \cos(\vartheta)\right]$$

it is clear that the interior of the parentheses have to be equal, so

$$\left[\tan^2(\lambda) + \tan^2(\mu) - 2 \cdot \tan(\lambda) \cdot \tan(\mu) \cdot \cos(\vartheta)\right] = \left[\sec^2(\lambda) + \sec^2(\mu) - 2 \cdot \sec(\lambda) \cdot \sec(\mu) \cdot \cos(\omega)\right]$$

which is the same as

$$-2\cdot\tan(\lambda)\cdot\tan(\mu)\cdot\cos(\vartheta)=\sec^2(\lambda)-\tan^2(\lambda)+\sec^2(\mu)-\tan^2(\mu)-2\cdot\sec(\lambda)\cdot\sec(\mu)\cdot\cos(\omega)$$

that can also be expressed as

$$-2\cdot\tan(\lambda)\cdot\tan(\mu)\cdot\cos(\vartheta)=\frac{1}{\cos^2(\lambda)}-\frac{\sin^2(\lambda)}{\cos^2(\lambda)}+\frac{1}{\cos^2(\mu)}-\frac{\sin^2(\mu)}{\cos^2(\mu)}-2\cdot\sec(\lambda)\cdot\sec(\mu)\cdot\cos(\omega)$$

that if we remember that

$$\sin^2+\cos^2=1$$

, is exactly the same as

$$-2\cdot\tan(\lambda)\cdot\tan(\mu)\cdot\cos(\vartheta)=1+1-2\cdot\sec(\lambda)\cdot\sec(\mu)\cdot\cos(\omega)$$

which was the equation in the math-notation deduction.

□

And now, if we rearrange the previous equation we get to the cosine-formula

$$\cos(\omega)=\cos(\mu)\cdot\cos(\lambda)+\sin(\mu)\cdot\sin(\lambda)\cdot\cos(\vartheta)$$

□

And because the problem is symmetric, in the same way we have

$$\cos(\omega)=\cos(\mu)\cdot\cos(\lambda)+\sin(\mu)\cdot\sin(\lambda)\cdot\cos(\vartheta)$$

we can write

$$\cos(\mu)=\cos(\lambda)\cdot\cos(\omega)+\sin(\lambda)\cdot\sin(\omega)\cdot\cos(\eta)$$

and

$$\cos(\lambda)=\cos(\omega)\cdot\cos(\mu)+\sin(\omega)\cdot\sin(\mu)\cdot\cos(\kappa)$$

So for any three given points there're three possible cosine-formulas.

The sine-formula

Math Notation

$$\cos(\omega) = \cos(\mu)\cdot\cos(\lambda) + \sin(\mu)\cdot\sin(\lambda)\cdot\cos(\vartheta)$$

$$\therefore$$

$$\cos^2(\omega) + \cos^2(\mu)\cdot\cos^2(\lambda) - 2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda) = \sin^2(\mu)\cdot\sin^2(\lambda)\cdot\cos^2(\vartheta)$$

$$\therefore$$

$$\cos^2(\omega) + \cos^2(\mu)\cdot\cos^2(\lambda) - 2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda)$$
$$=$$
$$1 - \cos^2(\mu) - \cos^2(\lambda) + \cos^2(\mu)\cdot\cos^2(\lambda) - \sin^2(\mu)\cdot\sin^2(\lambda)\cdot\sin^2(\vartheta)$$

$$\therefore$$

$$\sin^2(\vartheta) = \frac{1 - \cos^2(\mu) - \cos^2(\lambda) - \cos^2(\omega) + 2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda)}{\sin^2(\mu)\cdot\sin^2(\lambda)}$$

$$\therefore$$

$$\frac{\sin(\vartheta)}{\sin(\omega)} = \sqrt{\frac{1 - \cos^2(\mu) - \cos^2(\lambda) - \cos^2(\omega) + 2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda)}{\sin^2(\mu)\cdot\sin^2(\lambda)\cdot\sin^2(\omega)}}$$

$$\therefore$$

$$\frac{\sin(\vartheta)}{\sin(\omega)} = \frac{\sin(\eta)}{\sin(\mu)} = \frac{\sin(\kappa)}{\sin(\lambda)}$$

$$\square$$

Explanation

Once again, in case you need it, in this section we will explore in detail the previous deduction, so that the path to the sine-formula it's clear and the formula itself is understandable. The starting point of the deduction,

$$\cos(\omega) = \cos(\mu)\cdot\cos(\lambda) + \sin(\mu)\cdot\sin(\lambda)\cdot\cos(\vartheta)$$

, is just the cosine-formula, and we already know how to get to it. (Keep in mind that for any three given points there're three possible cosine-formulas, what follows next could be done for any of those three.) Let's take a look at the next step:

$$\cos^2(\omega) + \cos^2(\mu)\cdot\cos^2(\lambda) - 2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda) = \sin^2(\mu)\cdot\sin^2(\lambda)\cdot\cos^2(\vartheta)$$

This step is the result of applying to simple modifications to the starting point. First we'll take

$$\cos(\omega) = \cos(\mu)\cdot\cos(\lambda) + \sin(\mu)\cdot\sin(\lambda)\cdot\cos(\vartheta)$$

and rearrange it as

$$\cos(\omega) - \cos(\mu)\cdot\cos(\lambda) = \sin(\mu)\cdot\sin(\lambda)\cdot\cos(\vartheta)$$

Then we will square both sides

$$\left(\cos(\omega) - \cos(\mu)\cdot\cos(\lambda)\right)^2 = \left(\sin(\mu)\cdot\sin(\lambda)\cdot\cos(\vartheta)\right)^2$$

and by solving the parentheses we get to the equation we where looking for

$$\cos^2(\omega) + \cos^2(\mu)\cdot\cos^2(\lambda) - 2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda) = \sin^2(\mu)\cdot\sin^2(\lambda)\cdot\cos^2(\vartheta)$$

□

Then, from that previous equation,

$$\cos^2(\omega) + \cos^2(\mu)\cdot\cos^2(\lambda) - 2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda)$$
$$=$$
$$1 - \cos^2(\mu) - \cos^2(\lambda) + \cos^2(\mu)\cdot\cos^2(\lambda) - \sin^2(\mu)\cdot\sin^2(\lambda)\cdot\sin^2(\vartheta)$$

is deduced. The left-side hasn't change, so we will just focus on what happened to the right-side.

In

$$\sin^2(\mu)\cdot\sin^2(\lambda)\cdot\cos^2(\vartheta)$$

where applying the relation

$$\sin^2+\cos^2=1$$

obtaining

$$\sin^2(\mu)\cdot\sin^2(\lambda)\cdot\left(1-\sin^2(\vartheta)\right)$$

Solving the parenthesis, we get

$$\sin^2(\mu)\cdot\sin^2(\lambda)-\sin^2(\mu)\cdot\sin^2(\lambda)\cdot\sin^2(\vartheta)$$

where we will apply the same relation again

$$\left(1-\cos^2(\mu)\right)\cdot\left(1-\cos^2(\lambda)\right)-\sin^2(\mu)\cdot\sin^2(\lambda)\cdot\sin^2(\vartheta)$$

And solving the parenthesis again, we get to the right-side of the equation that we where trying to deduce

$$1-\cos^2(\mu)-\cos^2(\lambda)+\cos^2(\mu)\cdot\cos^2(\lambda)-\sin^2(\mu)\cdot\sin^2(\lambda)\cdot\sin^2(\vartheta)$$

□

So now, we know that

$$\cos^2(\omega)+\cos^2(\mu)\cdot\cos^2(\lambda)-2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda)$$
$$=$$
$$1-\cos^2(\mu)-\cos^2(\lambda)+\cos^2(\mu)\cdot\cos^2(\lambda)-\sin^2(\mu)\cdot\sin^2(\lambda)\cdot\sin^2(\vartheta)$$

But

$$\cos^2(\mu)\cdot\cos^2(\lambda)$$

is repeated in both sides, so we can get rid of it. After doing so, if solved for

$$\sin^2(\vartheta)$$

the equation looks like

$$\sin^2(\vartheta) = \frac{1-\cos^2(\mu)-\cos^2(\lambda)-\cos^2(\omega)+2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda)}{\sin^2(\mu)\cdot\sin^2(\lambda)}$$

□

Now, for a reason that may look clear later, to get to the next step we will divide both sides by

$$\sin^2(\omega)$$

and then do the square root of both sides to get to

$$\frac{\sin(\vartheta)}{\sin(\omega)} = \sqrt{\frac{1-\cos^2(\mu)-\cos^2(\lambda)-\cos^2(\omega)+2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda)}{\sin^2(\mu)\cdot\sin^2(\lambda)\cdot\sin^2(\omega)}}$$

□

There's only one step left, which requires a bit of reasoning:

We've done this whole process for the cosine-formula

$$\cos(\omega) = \cos(\mu)\cdot\cos(\lambda)+\sin(\mu)\cdot\sin(\lambda)\cdot\cos(\vartheta)$$

and we've obtained

$$\frac{\sin(\vartheta)}{\sin(\omega)} = \sqrt{\frac{1-\cos^2(\mu)-\cos^2(\lambda)-\cos^2(\omega)+2\cos(\omega)\cdot\cos(\mu)\cdot\cos(\lambda)}{\sin^2(\mu)\cdot\sin^2(\lambda)\cdot\sin^2(\omega)}}$$

If we had done it for

$$\cos(\mu) = \cos(\lambda)\cdot\cos(\omega)+\sin(\lambda)\cdot\sin(\omega)\cdot\cos(\eta)$$

we would have obtained

$$\frac{\sin(\eta)}{\sin(\mu)} = \sqrt{\frac{1-\cos^2(\lambda)-\cos^2(\omega)-\cos^2(\mu)+2\cos(\mu)\cdot\cos(\lambda)\cdot\cos(\omega)}{\sin^2(\lambda)\cdot\sin^2(\omega)\cdot\sin^2(\mu)}}$$

And if we had done it for

$$\cos(\lambda) = \cos(\omega)\cdot\cos(\mu)+\sin(\omega)\cdot\sin(\mu)\cdot\cos(\kappa)$$

we would have obtained

$$\frac{\sin(\kappa)}{\sin(\lambda)} = \sqrt{\frac{1-\cos^2(\omega)-\cos^2(\mu)-\cos^2(\lambda)+2\cos(\lambda)\cdot\cos(\omega)\cdot\cos(\mu)}{\sin^2(\omega)\cdot\sin^2(\mu)\cdot\sin^2(\lambda)}}$$

Take a closer look to the three of them. The right-part of all of them is exactly the same, hence, the left parts have to be the same two, this means that:

$$\frac{\sin(\vartheta)}{\sin(\omega)} = \frac{\sin(\eta)}{\sin(\mu)}$$

,

$$\frac{\sin(\eta)}{\sin(\mu)} = \frac{\sin(\kappa)}{\sin(\lambda)}$$

and

$$\frac{\sin(\vartheta)}{\sin(\omega)} = \frac{\sin(\kappa)}{\sin(\lambda)}$$

which are the three sine-formulas.

□